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LETTER TO THE EDITOR

Exact (2+1)-dimensional solutions for two discrete velocity Boltzmann models with four independent densities

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tial variables, relaxing towards non-uniform Maxwellian equilibrium states with two exponential variables. These are also solutions of another model with four independent densities and eight velocities oriented towards the eight corners of a cube. The positivity problem for the densities is non-trivial.

There is a continuous interest in the study of the discrete Boltzmann models, where the velocities can only take the discrete values $\boldsymbol{v}_i$, $|\boldsymbol{v}_i| = 1$, because people hope to find useful results for both kinetic theory and fluid mechanics. Since the popular Broadwell [1] model many others have been proposed [2]. To each velocity $\boldsymbol{v}_i$ is associated a density N_i and for the N_i with two coordinates x, y we must consider models with velocities in a plane or in a three-dimensional space.

The simplest solutions in one, two or three coordinate variables are the similarity shock waves. These are rational solutions with one exponential variable. It has recently been understood [3] that the $(1+1)$-dimensional (space x, time t) solutions are simply the sums of two such similarity waves and four classes of solutions were found: (i) $(1+1)$-dimensional shock waves [3], (ii) periodic solutions in space [3] propagating when the time is growing, (iii) periodic non-propagating solutions [3–6], (iv) densities N_i not relaxing towards constant Maxwellians [7].

For the $(2+1)$-dimensional solutions (space x, y, time t), although solutions of type (i) and (iv) have been obtained, only for class (iv) has the positivity problem of the N_i been entirely overcome. We present such positive solutions here.

We consider the square [2, 8] velocity model, sometimes attributed to Maxwell, with $\boldsymbol{v}_1$, $\boldsymbol{v}_3$ along the positive x and y axis, $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_3 + \boldsymbol{v}_4 = 0$, leading to the equations:

$$N_{1t} + N_{1x} = N_{2t} - N_{2x} = -N_{3t} - N_{3y} = -N_{4t} + N_{4y} = aN_3N_4 - N_1N_2 \qquad a > 0. \tag{1}$$

Equivalent equations are valid for a cubic [2] model with eight velocities oriented towards the eight corners of a cube and with four independent N_i ($N_6 = N_1$, $N_5 = N_2$, $N_8 = N_3$, $N_7 = N_4$)

$$N_{1t} + N_{1y} + N_{1x} = N_{2t} - N_{2y} - N_{2x} = -N_{3t} - N_{3y} + N_{3x} = -N_{4t} + N_{4y} - N_{4x} = aN_3N_4 - N_1N_2 \tag{1'}$$

which are reduced to (1) with the change $x + y = 2X$, $y - x = 2Y$.

For (1) the total mass $M=\Sigma N_i$ ($i=1,\ldots,8$ for (1')). Both mass and momentum conservation laws hold. For instance $M_t+\partial_x J_{(x)}+\partial_y J_{(y)}=0$ with momentum components $J_{(x)}=N_1-N_2$, $J_{(y)}=N_3-N_4$. For $a>0$ but $a\neq 1$, microreversibility is violated. Introducing [8] the relative entropy $H=\Sigma N_i \log(N_i/\alpha_i)$, $\alpha_i>0$, $\alpha_1\alpha_2=a\alpha_3\alpha_4$ we find from (1) as usual $H_t+\partial_x\ldots+\partial_y\ldots\leqslant 0$. The similarity shock waves are

$$N_i=n_{0i}+n_iD^{-1} \qquad D=1+d\exp(\gamma x+\tau y+\rho t) \tag{2}$$

n_{0i}, n_i, d, ρ, γ, τ being constants. The (2+1)-dimensional solutions are simply the superposition of such solutions:

$$N_i=n_{0i}+\Sigma_j n_{ji}D_j^{-1} \qquad D_j=1+d_j\exp(\gamma_j x+\tau_j y+\rho_j t) \tag{3}$$

with $j=1,2,3$ if the components are real while $j=1,2,3,4$ if the components are two by two complex conjugate and build periodic solutions.

We substitute (3) into (1) and write that the coefficients of the D_j^{-1}, D_j^{-2}, constant, $(D_mD_p)^{-1}$, $m\neq p$, terms are zero:

$$\begin{aligned} n_{j1}(\rho_j+\gamma_j) &= n_{j2}(\rho_j-\gamma_j)=-n_{j3}(\rho_j+\tau_j)=n_{j4}(\tau_j-\rho_j)=an_{j3}n_{j4}-n_{j1}n_{j2}\\ &=-a(n_{03}n_{j4}+n_{04}n_{j3})+n_{01}n_{j2}+n_{02}n_{j1}\\ n_{01}n_{02}=an_{03}n_{04} &\qquad a(n_{p3}n_{m4}+n_{p4}n_{m3})=n_{p1}n_{m2}+n_{p2}n_{m1} \qquad p\neq m. \end{aligned} \tag{4}$$

There exist 32 parameters and 31 relations for the periodic solutions $j=1,\ldots,4$. For the sum of three jth components, the number of parameters and relations is 25 and 19. In this last case, with at most six arbitrary parameters, we have numerically solved the system (4). Only for one subclass (non-uniform Maxwellians) have we entirely overcome the positivity problem for the N_i. This problem becomes simpler because one jth component, only t dependent, does not enter into the discussion. The positivity problem is reduced to a positive superposition of two components.

Subsequently we limit the study to $N_i(x,y,t)$ with non-uniform Maxwellians $N_i^M(x,y)$. In (3) the two first components are t independent ($\rho_1=\rho_2=0$) while the third one is t dependent ($\gamma_3=\tau_3=0$). From (4) we find $n_{j1}+n_{j2}=n_{j3}+n_{j4}=0$, $j=1,2$; $n_{31}=n_{32}=-n_{33}=-n_{34}$ and in (3):

$$\begin{aligned} &N_1^M=n_{01}+n_{11}D_1^{-1}+n_{21}D_2^{-1} \qquad N_3^M=n_{03}+n_{13}D_1^{-1}+n_{23}D_2^{-1}\\ &N_i^M+N_{i+1}^M=n_{0i}+n_{0i+1} \qquad i=1,3 \qquad D_j=1+\exp(\gamma_j x+\tau_j y)\\ &N_1-N_1^M=N_2-N_2^M=pD^{-1} \qquad N_3-N_3^M=N_4-N_4^M=-pD^{-1}\\ &D=1+d\exp(\rho t). \end{aligned} \tag{5}$$

For the third t-dependent component (in (5) we have put $\rho_3=\rho$, $n_{31}=p$), the remaining constraints (4) on the parameters being

$$\rho=p(a-1)=an_{43}^{+}+n_{21}^{+} \qquad n_{ij}^{\mp}=n_{0i}\mp n_{0j} \qquad a\neq 1 \tag{6}$$

ρ and p can be deduced once the n_{0i} are known. For $N_i^M>0$, we necessarily have $n_{0i}>0$ and in (6), $\rho>0$. Consequently $\lim pD^{-1}=0$ when $t\to\infty$; choosing $d>0$ large, $pD^{-1}<p(1+d)^{-1}$ can be arbitrarily small for $t\geqslant 0$ and for the study of $N_i>0$ we can restrict to $N_i^M>0$. Here $a\neq 1$ while $a=1$ will be allowed for the N_i^M. We notice that this time-dependent component does not give supplementary constraints on the N_i^M parameters (a property not true in general [3, 7]).

In the following, we determine the twelve n_{0i}, n_{j1}, n_{j3}, γ_j, τ_j, $j=1,2$, of the non-uniform Maxwellians and find sufficient conditions for the positivity. The results being analytical, the reader can check that positive (2+1)-dimensional solutions exist. There remains in (4) eight independent relations:

$$-\tau_j=\gamma_j n_{j1}/n_{j3}=-an_{43}^{-}+n_{j1}n_{21}^{-}/n_{j3}=-an_{j3}+n_{j1}^2/n_{j3}\qquad j=1,2 \tag{7}$$

$$an_{13}n_{23}=n_{11}n_{21}\qquad an_{03}n_{04}=n_{01}n_{02}. \tag{8}$$

From the n_{j1}, n_{j3} we can deduce the γ_j, τ_j. Furthermore, taking into account the condition $\gamma_1\tau_2\neq\gamma_2\tau_1$ ($\gamma_j x+\tau_j y$ must span a two-dimensional space) into (7), then the quadratic relations between n_{ji}, n_{0i} become linear:

$$n_{1i}+n_{2i}+n_{0i}=n_{0i+1}\qquad i=1,3. \tag{9}$$

Finally, putting aside γ_j, τ_j, we have eight parameters and four relations (8) and (9), leaving four arbitrary parameters chosen to be $(n_{01}, n_{02}, n_{03}, n_{11})$ and we must find the four others $(n_{04}, n_{21}, n_{13}, n_{23})$ so that $N_i^M>0$. The algebraic determination is simple: (8) and (9) give n_{04}, n_{21} and both the sum and the product of n_{13}, n_{23} lead to two possible determinations for n_{13}, n_{23}:

$$2z^{\pm}=n_{04}-n_{03}\pm\sqrt{\Delta}\qquad \Delta=(n_{04}-n_{03})^2-4n_{11}n_{21}/a\qquad n_{13}=z^{\pm}\qquad n_{23}=z^{\mp}. \tag{10}$$

The non-trivial problem is the positivity one. Taking into account $D_j^{-1}\geqslant 1$ in (5), sixteen constraints are sufficient for $N_i^M>0$:

$$n_{0i}>0\qquad n_{0i}+\Sigma_{j=1,2}n_{ji}>0\qquad n_{0i}+n_{ji}>0\qquad i=1,\dots,4;\quad j=1,2. \tag{11}$$

Our assumptions on the arbitrary parameters, sufficient for (11), are

$$n_{0i}>0\qquad i=1,2,3 \tag{12a}$$

$$0<n_{11}<n_{02}<n_{01}. \tag{12b}$$

From (8)-(12a), $n_{04}>0$ and the first four (11) conditions are satisfied. Recalling $n_{j1}+n_{j2}=n_{j3}+n_{j4}=0$, the four following ones in (11) are consequences of (9). It remains for the last eight ones in (11) to be rewritten:

$$-n_{0i}<n_{ji}<n_{0i+1}\qquad j=1,2;\quad i=1,3. \tag{11$'$}$$

The inequalities (11′) for n_{11} is a consequence of the assumption (12a), while from (9) the n_{21} ones are then deduced. We rewrite (10):

$$n_{04}-z^{\pm}=n_{03}+z^{\mp}=(n_{03}+n_{04}\mp\sqrt{\Delta})/2. \tag{10$'$}$$

Then all the last (11′) inequalities for n_{13}, n_{23} are satisfied if the last term in (10′) is positive or $\sqrt{\Delta}<n_{03}+n_{04}$. For this ultimate result ensuring $N_i^M>0$, we first notice that, due to (9)-(12b), $n_{21}<0$ or $\Delta>0$ and real z exist in (10). Second we establish a set of inequalities: $-n_{11}n_{21}=n_{11}(n_{11}-n_{02}+n_{01})<n_{11}n_{01}<n_{02}n_{01}\rightarrow -n_{21}n_{11}/a<n_{03}n_{04}\rightarrow\Delta<(n_{03}+n_{04})^2$ (see table 1 for a summary of the results).

In conclusion, $N_i^M(x,y)>0$ and in (2+1) dimensions $N_i(x,y,t)>0$ exist. Further, for the N_i^M, the $a>0$ parameter does not enter the assumption (12a,b) and the algebraic determination is valid for $a=1$. These Maxwellians exist whether the microreversibility is violated or not. In table 1, we give the values for $n_{11}=n_{03}=1$, $n_{02}=2$, $n_{01}=3$.

Table 1. Parameters for N_i^M, N_i equation (5).

General results: free parameters n_{0i}, $i=1,2,3$; n_{11}			
N_i^M: $n_{04}=n_{01}n_{02}(an_{03})^{-1}$	$n_{21}=n_{02}-n_{11}-n_{01}$		
$2z^{\pm}=n_{04}-n_{03}\pm[(n_{04}-n_{03})^2-4n_{11}n_{21}/a]^{1/2}$			
$n_{ij}^{\pm}=n_{0i}\pm n_{0j}$	$\tau_j=an_{43}^{-}-n_{21}^{-}n_{j1}/n_{j3}$	$\gamma_j=-\tau_j n_{j1}/n_{j3}$	$j=1,2$
N_i: $a\neq 1$	$\rho=an_{43}^{+}+n_{21}^{+}$	$p=\rho/(a-1)$	
Example: $n_{03}=n_{11}=1$	$n_{02}=2$	$n_{01}=3$	
N_i^M: $n_{04}=6/a$	$n_{21}=-2$	$2z^{\pm}=-1+6/a\pm[(1-6/a)^2+8/a]^{1/2}$	
$\gamma_1=-1+z^{\pm}(a-6)$			
$\gamma_2=z^{\pm}(6-a)/2-1$	$\tau_1=6-a+1/z^{\pm}$	$\tau_2=6-a-2/z^{\mp}$	
N_i: $a\neq 1$	$\rho=11+a$	$p=(11+a)/(a-1)$	

While the total masses M^M and M are constants for both the non-uniform Maxwellian and the exact solution, in contrast the momentum $\boldsymbol{J}^M$ and $\boldsymbol{J}$ are non-uniform in the space:

$$M=M^M=\Sigma n_{0i} \qquad J_{(x)}=J_{(x)}^M=n_{01}-n_{02}+2\Sigma n_{j1}D_j^{-1}$$
$$J_{(y)}=J_{(y)}^M=n_{03}-n_{04}+\Sigma n_{j3}D_j^{-1}. \tag{13}$$

Linearising around the non-uniform Maxwellians (5)-(12a, b) and assuming small t-dependent perturbations, an exact linearised solution exists. We define $N_i^L=N_i^M+\delta N_i$, $\delta N_i=\delta_i\exp(\mu t)$, substitute into (1) and find at the linear approximation level: $\delta_1=\delta_2=-\delta_3=-\delta_4$ while $\mu=-[a(n_{03}+n_{04})+n_{01}+n_{02}]$ is a negative eigenvalue.

Finally we notice the existence of another class of non-uniform Maxwellians which are periodic in x but not in y. Starting with

$$N_i^M=n_{0i}+2\,\mathrm{Re}(n_{iR}+\mathrm{i}n_{iI})D^{-1} \qquad N_i^M+N_{i+1}^M=n_{0i}+n_{0i+1} \qquad i=1,3$$
$$D=1+d\exp[(\tau_R+\mathrm{i}\tau_I)y+\mathrm{i}\gamma_I x] \tag{14}$$

(subscripts R and I for real and imaginary parts) and substituting into (1) we find

$$n_1\gamma=-n_3\tau=-an_3^2+n_1^2=-an_3n_{43}^{-}+n_1n_{21}^{-} \qquad a|n_3|^2=|n_1|^2$$
$$an_{03}n_{04}=n_{01}n_{02}. \tag{15}$$

For the resolution we define $n_1/n_3=\sqrt{a}\exp(\mathrm{i}z)$ and find

$$\cos z=n_{21}^{-}(\sqrt{a}\,n_{43}^{-})^{-1} \qquad \tau_R=[(n_{43}^{-})^2a-(n_{21}^{-})^2]/n_{43}^{-}. \tag{16a}$$

There are three arbitrary parameters, chosen to be $n_{0i}>0$, $i=1,2,3$; then $n_{04}>0$ is given by (15), we require $|\cos z|<1$ in (16a) and trivial calculations lead to the solution:

$$N_1^M=n_{01}+n_{21}^{-}\,\mathrm{Re}(1+\mathrm{i}\,\mathrm{tg}\,z)D^{-1} \qquad N_3^M=n_{03}+n_{43}^{-}\,\mathrm{Re}\,D^{-1}$$
$$D=1+d\exp\tau_R \qquad [y(1-\mathrm{i}\cot z)+\mathrm{i}x(\sin z)^{-1}a^{-1/2}]. \tag{16b}$$

In order to avoid poles for the N_i (or zeros for D), we must limit the solutions to half-planes, i.e. x and semi-axis $y>0$ or <0 such that either $|d|\exp\tau_R y>1$ (for $\tau_R y\geq 0$) or <1 ($\tau_R y\leq 0$). Choosing $|d|$ sufficiently large in the first case and sufficiently small in the second, one can show that the positivity of the N_i is satisfied in these half-planes. These positivity results come from the fact that lim $N_i = n_{0i}$ when $|d| \to \infty$ while when $|d| \to 0$, lim $N_i = n_{0i+1}$, $i = 1, 3$, and lim $N_i = n_{0i-1}$, $i = 2, 4$. For instance, $N_i > n_{0i} - |n_{43}^-|/(|d|-1)$, $j = 3, 4$ in the first case while in the second case $N_3 > n_{03}$ if $n_{43}^- > 0$ and $N_3 > n_{03} - (n_{03} - n_{04})/(1-|d|)$ if $n_{43}^- < 0$.

A first class of positive exact (2+1)-dimensional solutions (here relaxing towards non-uniform Maxwellians) has been constructed for the first time. It seems worth comparing these Maxwellians (see also [2]) with the equivalent ones in one spatial dimension [7]. For the planar models with microreversibility satisfied, they do not exist when the number of velocities is less than ten; the result implying that we need sufficient degrees of freedom for their existence. Here, adding one spatial coordinate (another way to open new degrees of freedom) they still exist for the square model. For the macroscopic total mass and momentum associated with their time-dependent solutions, they were constants in 1+1 dimensions while here the momentum is non-uniform in space. I hope to be able to tackle the positivity difficulty so that other classes could be obtained.

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